

Canonic form of linear quaternion functions

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Abstract

The general linear quaternion function of degree one is a sum of terms with quaternion coefficients on the left and right. The paper considers the canonic form of such a function, and builds on the recent work of Todd Ell, who has shown that any such function may be represented using at most four quaternion coefficients. In this paper, a new and simple method is presented for obtaining these coefficients numerically using a matrix approach which also gives an alternative proof of the canonic forms.

1 Introduction

In a recent paper [1], Todd Ell has shown that a linear quaternion function of the form:

$$f(q) = \sum_{p=1}^P m_p q n_p \quad (1)$$

where the coefficients m_p and n_p , and the variable q , are quaternions, can be expressed in a canonic form using a maximum of four quaternion coefficients, irrespective of the number of terms P . Ell's paper gives a method to obtain the four coefficients of the canonic form algebraically without explicit decomposition of the original quaternion coefficients m_p and n_p into their four components. This paper does not improve on that result, but it does present a simple method for obtaining the four coefficients of the canonic form numerically. The canonic form is¹:

$$f(q) = Aq + Bqi + Cqj + Dqk \quad (2)$$

Subsequent to the publication of this result by Todd Ell, a paper by Littlewood in 1931 was found to contain this canonic form [3] although Littlewood gave no justification for his statement that it was 'general' (assumed to mean 'canonic') nor a method to obtain the four coefficients from the general case as in equation 1. No earlier papers are currently known that consider this form. In a later paper with Richardson, the following equation appeared [4, page 333]:

$$a(x) = \sum_{q=1}^n a_q x e_q = \sum_{p,q=1}^n a_{pq} e_p x e_q$$

The context of this equation is non-commutative algebras in general. In the specific case of quaternions $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$, and the result given in equation 3 in the next section is a special case of Littlewood and Richardson's expression.

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¹It is also possible to obtain a similar form with the quaternion coefficients on the right and the quaternion operators i , j and k on the left, as was shown in [1].

2 Preliminaries

Consider a single term in equation 1: $m_p q n_p$ where we write $m_p = w_p + x_p \mathbf{i} + y_p \mathbf{j} + z_p \mathbf{k}$ and $n_p = w'_p + x'_p \mathbf{i} + y'_p \mathbf{j} + z'_p \mathbf{k}$. The product may be expanded as follows:

$$\begin{aligned} m_p q n_p &= (w_p + x_p \mathbf{i} + y_p \mathbf{j} + z_p \mathbf{k}) q (w'_p + x'_p \mathbf{i} + y'_p \mathbf{j} + z'_p \mathbf{k}) \\ &= \begin{aligned} &w_p w'_p q + w_p x'_p q \mathbf{i} + w_p y'_p q \mathbf{j} + w_p z'_p q \mathbf{k} + \\ &x_p w'_p \mathbf{i} q + x_p x'_p \mathbf{i} q \mathbf{i} + x_p y'_p \mathbf{i} q \mathbf{j} + x_p z'_p \mathbf{i} q \mathbf{k} + \\ &y_p w'_p \mathbf{j} q + y_p x'_p \mathbf{j} q \mathbf{i} + y_p y'_p \mathbf{j} q \mathbf{j} + y_p z'_p \mathbf{j} q \mathbf{k} + \\ &z_p w'_p \mathbf{k} q + z_p x'_p \mathbf{k} q \mathbf{i} + z_p y'_p \mathbf{k} q \mathbf{j} + z_p z'_p \mathbf{k} q \mathbf{k} + \end{aligned} \end{aligned} \quad (3)$$

We can immediately see that the full function in equation 1 can be written as a sum of sixteen summations over the products of components of the left and right quaternion coefficients:

$$\begin{aligned} f(q) = \sum_{p=1}^P m_p q n_p &= \begin{aligned} &\left(\sum_{p=1}^P w_p w'_p \right) q + \left(\sum_{p=1}^P w_p x'_p \right) q \mathbf{i} + \left(\sum_{p=1}^P w_p y'_p \right) q \mathbf{j} + \left(\sum_{p=1}^P w_p z'_p \right) q \mathbf{k} + \\ &\left(\sum_{p=1}^P x_p w'_p \right) \mathbf{i} q + \left(\sum_{p=1}^P x_p x'_p \right) \mathbf{i} q \mathbf{i} + \left(\sum_{p=1}^P x_p y'_p \right) \mathbf{i} q \mathbf{j} + \left(\sum_{p=1}^P x_p z'_p \right) \mathbf{i} q \mathbf{k} + \\ &\left(\sum_{p=1}^P y_p w'_p \right) \mathbf{j} q + \left(\sum_{p=1}^P y_p x'_p \right) \mathbf{j} q \mathbf{i} + \left(\sum_{p=1}^P y_p y'_p \right) \mathbf{j} q \mathbf{j} + \left(\sum_{p=1}^P y_p z'_p \right) \mathbf{j} q \mathbf{k} + \\ &\left(\sum_{p=1}^P z_p w'_p \right) \mathbf{k} q + \left(\sum_{p=1}^P z_p x'_p \right) \mathbf{k} q \mathbf{i} + \left(\sum_{p=1}^P z_p y'_p \right) \mathbf{k} q \mathbf{j} + \left(\sum_{p=1}^P z_p z'_p \right) \mathbf{k} q \mathbf{k} + \end{aligned} \end{aligned} \quad (4)$$

This expansion shows immediately that at least two canonic forms exist, as already shown in [1]. Grouping the four summations within the same column, each column sums to give a quaternion coefficient on the left of q :

$$f(q) = Aq + Bqi + Cqj + Dqk$$

An alternative is to group the four summations within the same row, where each row sums to give a quaternion coefficient on the right of q :

$$f(q) = qA' + i qB' + j qC' + k qD'$$

Now, a significant question is whether this is all, or whether there are additional canonic forms derivable from this expansion. More accurately, are there additional canonic forms with 4 quaternion coefficients (since a canonic form with more than 4 coefficients would be sub-optimal)? The next section recasts the problem in matrix form in order to show how this question might be answered.

In case this is thought by readers to be a trivial issue, Ljudmila Meister [5, §3.5] proposed the following as a general linear form: $f(q) = Aq + qB + CqD$. This has the correct number of quaternion coefficients (four, the same as the canonic form given by Littlewood in 1931 [3]) and it looks very plausible as a canonic form, since there can be only two terms with a single quaternion coefficient, one on the left and one on the right (any others could be trivially combined with these since $Aq + Eq = (A + E)q$), and one double-sided term with two coefficients. In fact, using the matrix formulation presented in the next section, it is argued that this form cannot be canonic. It is very hard to show this by an algebraic argument, but the matrix formulation presented in the next section makes it relatively simple.

3 Matrix and outer product formulation

The coefficients in equation 3 can be expressed as the following matrix, which is the *outer product* [6] of two vectors representing the left and right quaternion coefficients m_p and n_p . It follows from the fact that this matrix is the outer product of two vectors that it must be of rank 1 [6].

$$\begin{pmatrix} w_p w'_p & w_p x'_p & w_p y'_p & w_p z'_p \\ x_p w'_p & x_p x'_p & x_p y'_p & x_p z'_p \\ y_p w'_p & y_p x'_p & y_p y'_p & y_p z'_p \\ z_p w'_p & z_p x'_p & z_p y'_p & z_p z'_p \end{pmatrix} = \begin{pmatrix} w_p \\ x_p \\ y_p \\ z_p \end{pmatrix} \begin{pmatrix} w'_p & x'_p & y'_p & z'_p \end{pmatrix} \quad (5)$$

Similarly, we can express the coefficients in equation 4 as the following matrix. It follows from the fact that this matrix is 4×4 that it must be of rank 4 or less.

$$\mathbf{M} = \begin{pmatrix} \sum_{p=1}^P w_p w'_p & \sum_{p=1}^P w_p x'_p & \sum_{p=1}^P w_p y'_p & \sum_{p=1}^P w_p z'_p \\ \sum_{p=1}^P x_p w'_p & \sum_{p=1}^P x_p x'_p & \sum_{p=1}^P x_p y'_p & \sum_{p=1}^P x_p z'_p \\ \sum_{p=1}^P y_p w'_p & \sum_{p=1}^P y_p x'_p & \sum_{p=1}^P y_p y'_p & \sum_{p=1}^P y_p z'_p \\ \sum_{p=1}^P z_p w'_p & \sum_{p=1}^P z_p x'_p & \sum_{p=1}^P z_p y'_p & \sum_{p=1}^P z_p z'_p \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \quad (6)$$

\mathbf{M} may be decomposed into the sum of four matrices of rank 1, each of which can be factored into an outer product of two vectors as in equation 5. It follows that, for any number of terms P , equation 1 can be expressed as *at most* the sum of four terms, that is $P = 4$. The four rank 1 matrices which are summed to make \mathbf{M} can be obtained from a singular value decomposition [2], which can also yield the vectors whose outer product gives one of the components of \mathbf{M} . (Products of other pairs of columns will be zero because of orthogonality.) This can be done as follows:

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{L}\mathbf{V}^T$$

where \mathbf{U} and \mathbf{V} are orthogonal, and $\mathbf{\Sigma}$ is diagonal. Corresponding columns of matrices \mathbf{L} and \mathbf{V} are vectors whose outer products give the four matrices which sum to \mathbf{M} . These vectors are not unique, although the singular values in $\mathbf{\Sigma}$ are.

Clearly, a general linear quaternion function of the form in equation 1 with an arbitrary number of terms greater than 4 will yield a matrix \mathbf{M} of rank 4. It follows therefore that any such function can be expressed using at most 4 terms, where the left and right quaternion coefficients in each term are obtained from the outer product *factorisation* of rank 1 matrices composing \mathbf{M} (these rank 1 matrices can be obtained from a singular-value decomposition of \mathbf{M}):

$$f(q) = AqE + BqF + CqG + DqH$$

However, this is clearly not optimal, since it has eight quaternion coefficients, double the number needed in the canonic form given in equation 2. However, by decomposing matrix \mathbf{M} into the sum of four matrices each containing one column of \mathbf{M} , like this:

$$\mathbf{M} = \begin{pmatrix} m_{11} & 0 & 0 & 0 \\ m_{21} & 0 & 0 & 0 \\ m_{31} & 0 & 0 & 0 \\ m_{41} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & m_{12} & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & m_{32} & 0 & 0 \\ 0 & m_{42} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & m_{31} & 0 \\ 0 & 0 & m_{32} & 0 \\ 0 & 0 & m_{33} & 0 \\ 0 & 0 & m_{34} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & m_{41} \\ 0 & 0 & 0 & m_{42} \\ 0 & 0 & 0 & m_{43} \\ 0 & 0 & 0 & m_{44} \end{pmatrix} \quad (7)$$

we obtain the canonic form $f(q) = Aq + Bq\mathbf{i} + Cq\mathbf{j} + Dq\mathbf{k}$ in which the right-side quaternion coefficients reduce to the degenerate set $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and the left-side coefficients are given by $A = m_{11} + m_{21}\mathbf{i} + m_{31}\mathbf{j} + m_{41}\mathbf{k}$ and so on. (We could do the same with the rows of \mathbf{M} , in order to obtain the canonic form with $1, \mathbf{i}, \mathbf{j}$ and \mathbf{k} on the left.)

We now return to the ‘general’ form proposed by Ljudmila Meister [5, §3.5] and mentioned in the last paragraph of §2. This form is: $f(q) = Aq + qB + CqD$. It should be clear from the foregoing that the three terms in this function would yield a matrix \mathbf{M} of rank 3, and since the general case requires a matrix of rank 4, it is not canonic. It might appear that adding a fourth term (EqF) would fix the problem, since this would increase the rank to 4, but in fact this is not sufficient to yield a canonic form. This can be seen as follows. A corresponds to a column of the matrix \mathbf{M} , and B to a row. This leaves three other columns and rows, which in the general case, will be of rank 3. It follows that two double-sided terms like CqD and EqF are not sufficient to represent the rest of the terms of a general linear function, since they cannot yield a component of \mathbf{M} of rank greater than 2, and a rank of 3 is needed. We are thus led to conclude that a function of the form $f(q) = Aq + qB + CqD + EqF + GqH$

is required, which is again not optimal, since it has 8 quaternion coefficients. In fact, it is not necessary for all the terms in this function to be full quaternions, but as we now show, this does not improve on the form in equation 2.

If we take the first column and the first row for A and B , we would be left with a 3×3 matrix, which in the general case would be of rank 3. Factorising it into outer products we would need three terms with a pure quaternion on each side. Noting also that it is not necessary for both of A and B to be full quaternions, a ‘canonic’ form appears to be: $f(q) = Aq + qb + v_1qv_2 + v_3qv_4 + v_5qv_6$, where b and the v_i are vectors (pure quaternions). The number of real coefficients in this function is 25, 9 more than in the canonic forms. However, when factorising the 3×3 matrix within M we can factorise by columns, thus reducing the form to $f(q) = Aq + qb + v_1q\mathbf{i} + v_3q\mathbf{j} + v_5q\mathbf{k}$ which has 16 real coefficients and is therefore canonic, although it is just a variation of the form derived recently by Ell, and published by Littlewood in 1931.

It appears unlikely that there are any other decomposition(s) of M that would yield a canonic form with only 16 real coefficients.

4 Conclusion

In general, an arbitrary quaternion function of the form given in equation 1 cannot be reduced to less than four double-sided terms. That is, the function can always be reduced to a sum of 4 double-sided terms, with 8 quaternion coefficients. There are useful special cases where four of these coefficients are degenerate either on the left, or on the right (but not mixed) as already shown in [1].

The matrix formulation given in this paper provides a simple and systematic way to study the problem further, although it appears unlikely that the canonic form in equation 2 can be improved on.

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